

Generalizing the Kodama State I: Construction

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Abstract

The Kodama State is unique in being an exact solution to all the ordinary constraints of canonical quantum gravity that also has a well defined semi-classical interpretation as a quantum version of a classical spacetime, namely (anti)de Sitter space. However, the state is riddled with difficulties which can be tracked down to the complexification of the phase space necessary in its construction. This suggests a generalization of the state to real values of the Immirzi parameter. In this first part of a two paper series we show that one can generalize the state to real variables and the result is surprising in that it appears to open up an infinite class of physical states. We show that these states closely parallel the ordinary momentum eigenstates of non-relativistic quantum mechanics with the Levi-Civita curvature playing the role of the momentum. With this identification, the states inherit many of the familiar properties of the momentum eigenstates including delta-function normalizability. In the companion paper we will discuss the physical interpretation, CPT properties, and an interesting connection between the inner product and the Macdowell-Mansouri formulation of general relativity.

1 Introduction

Perturbative techniques in quantum field theory and their extension to quantum gravity are unparalleled in computational efficacy. In addition, because one can always retreat to the physical picture of particles as small field perturbations propagating on a classical background, perturbation theory maximizes the ease of transition from quantum to classical mechanics, and many processes can be viewed as quantum analogues of familiar classical events. However, the transparent physical picture disappears in systems where the distinction between background and perturbation to said background is blurred. Such systems include strongly interacting systems, such as QCD, or systems where there is no preferred background structure, such as general relativity. In contrast, non-perturbative and background independent approaches to quantum gravity do not distinguish background from perturbation, and are, therefore, appropriate for modeling the quantum mechanical ground state of the universe itself that, it is hoped, will serve as the vacuum on which perturbation theory can be based. However, this is often at the expense of losing the smooth transition from a quantum description to its classical or semi-classical counterpart as evidenced, for example, by the notorious problem of finding the low energy limit of Loop Quantum Gravity. The sticking point is that pure quantum spacetime may be sufficiently divorced from our classical understanding of fields on a smooth Riemannian manifold, that matching quantum or semi-classical states with classical analogues may be extremely difficult.

In this respect the Kodama state is unique. Not only is the state an exact solution to all the constraints of canonical quantum gravity, a rarity in itself, but it also has a well defined physical interpretation as the quantum analogue of a familiar classical spacetime, namely de Sitter or anti-de Sitter space depending on the sign of the cosmological constant[1, 2, 3]. Thus, the state is a candidate for the fulfillment of one of the distinctive advantages of a non-perturbative approach over perturbative techniques: the former has the potential to predict the purely quantum mechanical ground state on which perturbation theory can be based. In addition, the Kodama state has many beautiful mathematical properties relating the seemingly disparate fields of abstract knot theory and quantum field theory on a space of connections[4]. In particular, the exact form of the state is known in both the connection representation where it is the exponent of the Chern-Simons action, and in the q-deformed spin network representation where it is a superposition of

all spin networks with amplitudes given by the Kauffman bracket¹[6]. This connection played a pivotal role in the development of the loop approach to quantum gravity. One offshoot of the connection between the state and knot theory is that the relation with quantum groups allows for a reinterpretation of the role cosmological constant as the modulator of the deformation parameter of the quantum deformed group.

Ultimately, however, observation and experiment are the arbiters of the relevance of a physical theory, and cosmological evidence suggests that we live in an increasingly vacuum dominated universe, which is asymptotically approaching de Sitter space in the future as matter fields are diluted by the expansion of the universe, and possibly in the past as well as evidenced by the success of inflation models. Thus, the state with positive λ is particularly relevant to modern cosmology, and it opens up the possibility of making uniquely quantum mechanical predictions of a cosmological nature.

1.1 Problems

Despite all of these positive attributes of the Kodama state, the state is plagued with problems. Among these are the following:

- **Non-normalizability:** The Kodama state is not normalizable under the kinematical inner product, where one simply integrates $|\Psi|^2$ over all values of the complex Ashtekar connection. The state is not known to be normalizable under a physical inner product defined by, for example, path integral methods. Linearized perturbations around the state are known to be non-normalizable under a linearized inner product[7].
- **CPT Violation:** The states are not invariant under CPT[8]. This is particularly poignant objection in view of the CPT theorem of perturbative quantum field theory, which connects CPT violation with Lorentz violation. It is not known if the result carries over to non-perturbative quantum field theory, but it has yet to be demonstrated that the Kodama state does not predict Lorentz violation.

¹The loop transform is well understood and rigorous at the level of mathematical physics for Euclidean signature spacetime. For Lorentz signature spacetime, the loop transform is believed to be the Kauffman bracket, but the proof requires integrating along a real contour in the complex plane, and it is not as rigorous as in the real case (see e.g. [5]). The de Sitter state that we will present shares loop transform properties in common with the Euclidean signature Kodama state, so it is well defined in the loop basis.

- **Negative Energies:** It has been argued by analogy with a similar non-perturbative Chern-Simons state of Yang-Mills theory that the Kodama state necessarily contains negative energy sectors[8]. If the energy of one sector of the state is strictly positive, the CPT inverted state will necessarily contain negative energy sectors.
- **Non-Invariance Under Large Gauge Transformations:** Although the state is invariant under the small gauge transformations generated by the quantum constraints, it is not invariant under large gauge transformations where it changes by a factor related to the winding number of the map from the manifold to the gauge group. However, it has been argued that the non-invariance of the Kodama state under large gauge transformations give rise to the thermal properties of de Sitter spacetime²[9]. Thus, non-invariance under large gauge transformations could be a problem or a benefit, but it is deserving of mention.
- **Reality Constraints:** The Lorentzian Kodama state is a solution to the quantum constraints in the Ashtekar formalism where the connection is complex. To obtain classical general relativity one must implement reality conditions which ensure that the metric is real. It is an open problem as to how to implement these constraints on a general state. Generally it is believed that the physical inner product will implement the reality constraints, but this could change the interpretation of the state considerably.

1.2 Resolution

Many of the above problems can be tracked down to the complexification of the phase space necessary in the construction of the state. To see this, one can simply appeal to the Euclidean version of the state. In the Euclidean formalism, the gauge group $SO(4)$ splits into two left and right pieces as in the complex theory. Choosing the left handed part of the group, the canonical variables in the Ashtekar formalism consist of a *real* $SO(3)$ connection and its *real* conjugate momentum. The analogous state in the Euclidean theory

²Paradoxically, we will argue the opposite: that demanding invariance of the generalized states we will present under large gauge transformations gives rise to evidence of cosmological horizons, which in turn should give rise to the thermal nature of de Sitter space.

is a pure phase since the connection is real:

$$\Psi[A] = \mathcal{N} e^{-i \frac{3}{4k\lambda} \int Y_{CS}[A]}. \quad (1)$$

Although the state may not be strictly normalizable, one might expect that it is delta-function normalizable because it is pure phase. In fact, it has been shown that linearized perturbations to the Euclidean state are delta-function normalizable under a linearized inner product[7]. In addition, the state is CPT invariant due to the factor of i in the argument which inverts under time reversal canceling the action of parity. Although it is not known if the state has negative energies, one cannot use the standard argument that a positive energy sector will become a negative energy sector under CPT reversal, because the action of CPT is now trivial. Since the state is now pure phase, the level of the Chern-Simons theory is real. Thus, by fine tuning Newton's constant or the cosmological constant (within observational error), one can make the level an integer, in which case the state is invariant under large gauge transformations. Finally, there are no reality conditions in the Euclidean theory since the connection and its conjugate momentum are real. Thus, the Euclidean state appears to be free of most of the known problems associated with the Lorentzian state. However, the real world is Lorentzian: *can one salvage the Lorentzian Kodama state despite all these problems?*

The above properties of the Euclidean state suggest that the problems associated with the Lorentzian Kodama state are rooted in the complexification of the phase space. The phase space is complex because of a particular choice for a free parameter, the Immirzi parameter β , which is chosen to be the unit imaginary, $-i$, in the complex Ashtekar formalism. Modern formulations of Loop Quantum Gravity assume that β is an arbitrary *real* number[10]. The parameter is currently believed to be fixed by demanding consistency with the spin network derivation of the entropy of an isolated horizon, and Hawking's formula for the entropy of a static, spherically symmetric black hole[11]. The first few sections of this paper will be devoted to generalizing the state to real values of the Immirzi parameter. The discussion will initially follow along the the lines of [12], and then will diverge, addressing some deficiencies of that initial attempt at generalizing the Kodama state. We will show that generalizing the state opens up a large Hilbert space of states each parameterized by a particular configuration of the three-dimensional Riemannian curvature. By exploiting an analogy between these states and the ordinary momentum eigenstates of single particle quantum mechanics we will show that the states are delta function normalizable and orthogonal unless they

are parameterized by the same 3-curvature modulo $SU(2)$ gauge and diffeomorphism transformations. Using this property we will show that the states can be used to construct a natural Levi-Civita curvature operator. When this operator is used in the Hamiltonian constraint, all of the states are annihilated by the constraint. In a follow-up paper we will then show that the generalized states are free of most of the problems associated with the original incarnation of the Kodama state, and we will discuss the physical interpretation of the new states and their relation to de Sitter space. We conclude with an intriguing relation between the physical inner product of two generalized states and the Macdowell-Mansouri formulation of gravity.

2 Chiral Asymmetric Extension of the Kodama State

2.1 Chirally Asymmetric Gravity

Following along the lines of [12], we begin the construction of the states using a chirally asymmetric, complex action. This will allow us to make headway in generalizing the state to arbitrary *imaginary* values of the Immirzi parameter. Later we will analytically extend the states to real values of the Immirzi parameter. The starting point for the construction of the generalized Kodama states is the Holst action with a cosmological constant³[13]:

$$S_H = \frac{1}{k} \int_M \star e \wedge e \wedge R + \frac{1}{\beta} e \wedge e \wedge R - \frac{\lambda}{3} \star e \wedge e \wedge e \wedge e. \quad (2)$$

Here $e = \frac{1}{2}\gamma_I e^I$ is the frame field, $R = \frac{1}{4}\gamma_{[I}\gamma_{J]}R^{IJ}$ is the curvature of the $Spin(3,1)$ connection $\omega = \frac{1}{4}\gamma_{[I}\gamma_{J]}\omega^{IJ}$, $k = 8\pi G$, and $\star = -i\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$. The parameter β is the Immirzi parameter, which can be interpreted as the measure of parity violation built into the framework of quantum gravity.

³Throughout the paper we will work with a Lorentzian metric with signature $\eta = \text{diag}(-1, 1, 1, 1)$. The metric volume form ϵ_{IJKL} is defined such that $\epsilon_{0123} = -\epsilon^{0123} = +1$. Upper case Roman indices $\{I, J, K, L, \dots\}$ represent spacetime indices in the adjoint $Spin(3,1)$ representation space and range from 0 to 3. Lower case Roman indices $\{i, j, k, \dots\}$ are three dimensional indices in the adjoint representation of $SU(2)$, and range from 1 to 3. In the base manifold, spacetime indices are represented by Greek letters $\{\mu, \nu, \alpha, \beta, \dots\}$, and spatial indices are represented by lower case Roman indices in the beginning of the alphabet $\{a, b, c, \dots\}$.

The parameter does not affect the equations of motion in the absence of matter, and matter fields can be appropriately modified so that the resulting equations of motion reproduce that of the Einstein-Cartan action [14]. The parameter does play a significant role in the quantum theory where it fine tunes the scale where Planck scale discreteness occurs. At this stage we will take the parameter to be purely imaginary and later analytically extend to real values. The reason we begin with imaginary β is because in this case the Holst action splits into two independent left and right handed components. In this sense, imaginary values of β can not only be interpreted as a measure of parity violation, but more specifically they measure the degree of *chiral* asymmetry built into the framework of gravity. To see this, we introduce the left and right handed chiral projection operators $P_{L/R} = \frac{1}{2}(1 \mp i\star)$, and define the chirally asymmetric Einstein-Cartan action (writing $\Sigma \equiv e \wedge e$):

$$\begin{aligned}
S &= \frac{1}{k} \int_M 2(\alpha_L P_L + \alpha_R P_R) \star \Sigma \wedge (R - \frac{\lambda}{6} \Sigma) \\
&= \frac{2}{k} \int_M \alpha_L \star \Sigma_L \wedge (R_L - \frac{\lambda}{6} \Sigma_L) + \alpha_R \star \Sigma_R \wedge (R_R - \frac{\lambda}{6} \Sigma_R) \\
&= \frac{1}{k} \int_M (\alpha_L + \alpha_R) \star \Sigma \wedge (R - \frac{\lambda}{6} \Sigma) + i(\alpha_L - \alpha_R) \Sigma \wedge R. \tag{3}
\end{aligned}$$

The last line is the Holst action if we make the identifications $\alpha_L + \alpha_R = 1$ and

$$\beta = \frac{-i}{\alpha_L - \alpha_R}. \tag{4}$$

We note that in the limiting case when $\alpha_R = 0$ and $\beta = -i$ we recover the left handed Einstein-Cartan action whose phase space consists of the complex left-handed Ashtekar action and its conjugate momentum. The advantage of the this formalism is that the action splits into two components that, prior to the implementation of reality constraints, can be treated independently. The reality constraint requires that e^I and ω^{IJ} are real. This implies the constraints $\Sigma_L^{IJ} = \overline{\Sigma_R^{IJ}}$ and $\omega_L^{IJ} = \overline{\omega_R^{IJ}}$. We will proceed to construct the quantum constraints and a generalization of the Kodama state initially assuming all left handed variables are independent of right handed variables. Later we will impose the above reality constraints.

Proceeding to construct the constraints assuming left and right handed variables are independent we find that the constraint algebra splits into two independent copies which differ by handedness and by the relative coupling

constants α_L and α_R . We demand that the manifold has topology $\mathbb{R} \times \Sigma$ where Σ is the spatial topology. Introducing a monotonic time function t , we define a timelike vector field \bar{t} to be the canonical dual of the one form dt , so that $dt(\bar{t}) = 1$. We further split the vector field into components normal and parallel to the 3-manifold which we denote $\bar{t} = N\bar{n} + \bar{N}$. The vector field \bar{n} is the unit normal to Σ , N is the lapse, and \bar{N} is the shift. We will partially fix the gauge to the time gauge where $e_a^0 = 0$ with $a = \{1, 2, 3\}$ components in the base manifold. This is achieved by fixing the direction of the unit normal in the fibre so that $e^I(\bar{n}) = n^I = (1, 0, 0, 0)$. This is not strictly necessary in the Ashtekar formalism since self dual $spin(3, 1)$ variables in M can be pulled back to complex $su(2)$ valued variables in Σ . However, we will work in the time gauge in order to make contact with the real Ashtekar-Barbero formalism where gauge fixing is necessary. The left and right handed connections pullback naturally to Σ to form the canonical position variable $A_L^{ij} = \omega^{ij} + iK^{ij}$ and $A_R^{ij} = \omega^{ij} - iK^{ij}$. Here $K^{ij} = \epsilon^{ij}_k K^k$ and K^i is the extrinsic curvature defined by $\phi^* Dn^I$ where ϕ^* is the pullback of the map from Σ to M . In our gauge, the extrinsic curvature is then $K_a^i = \omega^i_{0a}$. The canonical momenta to ω_L^{ij} and ω_R^{ij} are the two forms $-\frac{i\alpha_L}{2k}\Sigma_L^{ij}$ and $\frac{i\alpha_R}{2k}\Sigma_R^{ij}$. Eventually we will want $\Sigma_L^{ij} = \Sigma_R^{ij} = E^i \wedge E^j$ where $E_a^i \equiv e_a^i$ is the spatial triad in the time gauge, but for now we are treating the two as independent. With this, the canonical commutation relations are

$$\begin{aligned} \{A_L^{ij}|_P, \Sigma_{kl}^L|_Q\} &= -\frac{2k}{\alpha_L} \delta_m^{[i} \delta_n^{j]} \delta(P, Q) \\ \{A_R^{ij}|_P, \Sigma_{kl}^R|_Q\} &= \frac{2k}{\alpha_R} \delta_m^{[i} \delta_n^{j]} \delta(P, Q) \\ \{A_L^{ij}|_P, \Sigma_{kl}^R|_Q\} &= \{A_R^{ij}|_P, \Sigma_{kl}^L|_Q\} = 0 \end{aligned} \quad (5)$$

Each of the constraints contain two independent left and right handed components:

$$C_H(N) = \alpha_L \int_{\Sigma} N (*\Sigma_{ij}^L \wedge (R_L^{ij} - \frac{\lambda}{3}\Sigma_L^{ij})) + (L \rightarrow R) \quad (6)$$

$$C_G(\lambda_L, \lambda_R) = \alpha_L \int_{\Sigma} D_L \lambda_{ij}^L \wedge \Sigma_L^{ij} - (L \rightarrow R) \quad (7)$$

$$C_D(\bar{N}) = \alpha_L \int_{\Sigma} \mathcal{L}_{\bar{N}} A_L^{ij} \wedge \Sigma_{ij}^L - (L \rightarrow R) \quad (8)$$

The Hamiltonian constraint, C_H , generates time reparameterizations through the shift, N , the ‘‘Gauss’’ or ‘‘gauge’’ constraint, C_G , generates infinitesimal

$SU_L(2) \times SU_R(2)$ transformations with λ_L and λ_R as generators, and the diffeomorphism constraint, C_D , generates infinitesimal three-dimensional diffeomorphisms along the vector field \tilde{N} . We now need to promote the constraints to quantum operators. We will work in the connection representation where the momenta are functional derivatives:

$$\Sigma_{ij}^L = \frac{2k}{\alpha_L} \frac{\delta}{\delta \omega_L^{ij}} \quad \Sigma_{ij}^R = -\frac{2k}{\alpha_R} \frac{\delta}{\delta \omega_R^{ij}}. \quad (9)$$

Since the left and right handed variables are independent the Hilbert space also splits into two copies: $\mathcal{H}_R \times \mathcal{H}_L$. Thus we will look for solutions of this form. With the operator ordering given above, the constraints immediately admit the Kodama-like solution:

$$\Psi[A_L, A_R] = \mathcal{N} \exp \left[-\frac{3}{4k\lambda} \left(\alpha_L \int_{\Sigma} Y_{CS}[A_L] - \alpha_R \int_{\Sigma} Y_{CS}[A_R] \right) \right]. \quad (10)$$

where the $Y[A] = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$ is the Chern-Simons three-form and the implied trace is in the adjoint representation of $su(2)$. Here we have used the fundamental identity

$$\frac{\delta}{\delta A_{ij}} \int_{\Sigma} A^p_q \wedge dA^q_p + \frac{2}{3} A^p_q \wedge A^q_r \wedge A^r_p = -2F^{ij}. \quad (11)$$

We note that in the limit that $\alpha_L = 1$ and $\alpha_R = 0$, we regain the original form of the Kodama state.

2.2 Imposing the Reality Constraints

We now need to impose the reality constraints $\Sigma_L = \overline{\Sigma_R}$ and $A_L = \overline{A_R}$. Imposing the constraints on the position variables is easy since these are just multiplicative operators. We define the real and imaginary parts of A_L by⁴

$$\omega^{ij} \equiv \text{Re}(A_L) = \frac{1}{2}(A_L + A_R) \quad (12)$$

$$K \equiv \text{Im}(A_L) = \frac{1}{2i}(A_L - A_R). \quad (13)$$

⁴Having fixed our index conventions in the previous sections, in the remaining sections we will drop all indices. Unless stated otherwise, we will work in the adjoint representation of $SU(2)$.

It then follows that $A_L = Re(A_L) + iIm(A_L) = \omega + iK$ and $A_R = \overline{A_L} = \omega - iK$. The constraint on the momentum variables is slightly more subtle due to the partial gauge fixing we have employed. Without gauge fixing we would have $\Sigma_L^{ij} = e^i \wedge e^j + i\epsilon^{ij}_k e^i \wedge e^0$, but in the time gauge $e_a^0 = 0$ so $\Sigma_L^{ij} = E^i \wedge E^j$ is real. To implement this in the quantum theory we define

$$\Sigma \equiv Re(\Sigma_L) = \frac{1}{2}(\Sigma_L + \Sigma_R) \quad (14)$$

$$C_\Sigma \equiv Im(\Sigma_L) = \frac{1}{2i}(\Sigma_L - \Sigma_R) = 0. \quad (15)$$

We now need to add the constraint C_Σ into the full set of constraints. We encounter a problem when evaluating the full set of commutators—the constraint algebra no longer closes. In particular, we find that the commutator between the Hamiltonian constraint and C_Σ yields a second class constraint proportional to the torsion of ω :

$$\{C_H, C_\Sigma\} \sim D_\omega * \Sigma = T. \quad (16)$$

Typically this second class constraint is solved at the classical level by replacing the unconstrained $SU(2)$ spin connection ω with the torsion-free Levi-Civita connection, $\Gamma = \Gamma[E]$, where Γ is a solution to the torsion condition $dE^i = -\Gamma^i_k \wedge E^k$. In our context this implies that the left and right spin connections are replaced by $A_L = \Gamma + iK$ and $A_R = \Gamma - iK$. With these replacements, the left and right handed connections will no longer commute: $\{\omega_L, \omega_R\} \neq 0$. This is our first indication that something will go wrong with this initial attempt at generalizing the Kodama state when the full set of constraints is employed. We will see that we can avoid this issue entirely by a proper reinterpretation of the problem.

However, there is another, potentially more serious problem associated with the introduction of the constraint C_Σ . In particular, the generalized state we have constructed does not satisfy the quantum constraint $C_\Sigma \Psi = 0$. To illustrate the problem it is useful to redefine the basis of our phase space such that $\Sigma = \frac{1}{2}(\Sigma_L + \Sigma_R)$ and C_Σ are the new canonical momenta up to numerical coefficients. The associated canonical position variables are

$$A_{-\frac{1}{\beta}} \equiv \alpha_L A_L + \alpha_R A_R = \Gamma + \frac{1}{\beta} K \quad (17)$$

$$A_\beta \equiv \frac{\alpha_L A_L - \alpha_R A_R}{\alpha_L - \alpha_R} = \Gamma - \beta K, \quad (18)$$

which can be seen from the canonical commutation relation that follow directly from 5,

$$\begin{aligned}
\{A_{-\frac{1}{\beta}}, C_\Sigma\} &= i2k \delta(P, Q) \\
\{A_\beta, \Sigma\} &= -i2k\beta \delta(P, Q) \\
\{A_{-\frac{1}{\beta}}, \Sigma\} &= 0 \\
\{A_\beta, C_\Sigma\} &= 0.
\end{aligned} \tag{19}$$

We recognize A_β and $\Sigma = E \wedge E$ as the Ashtekar-Barbero connection and its momentum that emerge in the real formulation of LQG. The reason for introducing these variables is that the constraint $C_\Sigma \Psi = 0$ takes a particularly simple form. In the connection representation, $C_\Sigma = 2k \frac{\delta}{\delta A_{-1/\beta}}$. Thus, we must have⁵

$$C_\Sigma \Psi = 2k \frac{\delta}{\delta A_{-1/\beta}} \Psi = 0 \quad \longrightarrow \quad \Psi = \Psi[A_\beta]. \tag{20}$$

That is, the wavefunction can only be a function of the Ashtekar-Barbero connection A_β and is independent of $A_{-1/\beta}$.

Now we need to check that the state (10) is only a function of A_β . To do so, we express the state in terms of the A_β and $A_{-1/\beta}$. We rewrite the state in a form that will be convenient for later use:

$$\Psi[A] = \mathcal{N} \exp \left[\frac{-3i}{4k\Lambda\beta^3} \int_\Sigma Y_{CS}[A] - (1 + \beta^2) Y_{CS}[\Gamma] + 2\beta(1 + \beta^2) \text{Tr}(K \wedge R_\Gamma) \right]. \tag{21}$$

Here Γ and K are explicit functions of both A_β and $A_{-1/\beta}$, given by

$$\Gamma = \frac{A_\beta + \beta^2 A_{-1/\beta}}{1 + \beta^2} \tag{22}$$

$$K = \frac{1}{\beta} (\Gamma - A_\beta). \tag{23}$$

We see that the state is explicitly a function of *both* A_β and $A_{-1/\beta}$. Thus, $C_\Sigma \Psi \neq 0$.

⁵The limiting case when $\beta \rightarrow \mp i$ must be treated separately here because in those cases we have an initial primary constraint that $\Sigma_{R/L} = 0$.

2.3 Resolution

The problems we have encountered with this initial attempt at generalizing the Kodama state are twofold. First, we encounter a second-class constraint whose solution requires that we introduce the torsion-free spin connection $\Gamma = \Gamma[E]$. This means that the left and right handed variables will no longer commute, or in the new variables, A_β and $A_{-1/\beta}$ will no longer commute. Second, we find that the reality constraint on the momentum requires that the wave function is a functional of A_β , which is not true for our left-right asymmetric state. We can recast the problem in a slightly more intuitive way by eliminating $A_{-1/\beta}$ in favor of the momentum Σ . That is, we explicitly write $A_{-1/\beta} = \frac{1}{\beta^2}((1 + \beta^2)\Gamma - A_\beta)$ and treat $\Gamma[E]$ as an explicit function of the momentum conjugate to A_β . Then the problem can be restated, *why is the wavefunction an explicit function of both position and momentum variables?* The problem of defining the commutator of A_β and $A_{-1/\beta}$ is transmuted into the problem of defining the operator $\Gamma[E]$ which occurs explicitly in the Hamiltonian through the Levi-Civita curvature, R_Γ , or the extrinsic curvature, $\frac{1}{\beta}(\Gamma - A)$, depending on how one writes the constraints. We will see that we can address both of these problems by analytically extending the state to real values of the Immirzi parameter, β . This will allow us to exploit an analogy between the generalized Kodama state and the non-relativistic momentum eigenstates, which will suggest a reinterpretation of the explicit momentum dependence of the state and at the same time suggest a natural definition of the Levi-Civita curvature operator R_Γ . This will be the subject of the rest of the paper.

3 The Generalized Kodama States

3.1 Properties of the real state

We now consider the state (21) when the Immirzi parameter β is taken to be a non-zero, but otherwise arbitrary real number. Modern formulations of Loop Quantum Gravity begin with arbitrary real values of β in the canonical construction because the analysis of real $SU(2)$ connections is better understood than that for complex connections. In addition, it is believed that thermodynamic arguments will eventually fix the value of the Immirzi parameter unambiguously. For our purposes, taking β to be real changes the properties of the generalized Kodama state considerably.

We first address the issue of the explicit momentum dependence of the state. We appeal to a similar situation in ordinary single particle quantum mechanics. The generalized Kodama for real values of β shares many properties in common with the ordinary momentum eigenstates. First of all, both states are pure phase. This means that they are bounded, which has implications for the inner product. Whereas the complex Kodama state is unbounded, which implies that the state is non-normalizable under a naive inner product, the real state is pure phase and therefore may be normalizable in the strict sense if the phase space is compact, or delta-function normalizable if the phase space is non-compact. Secondly, the momentum eigenstates share the property in common with the generalized Kodama state in that they ostensibly depend explicitly on both the momentum and position variables. Of course, the role of the momentum in the momentum eigenstates is very different from the role of the position variables. The state $\Psi_p(x) = \mathcal{N}e^{ip \cdot x - iEt}$ is explicitly a function of the position variable only, but it is *parameterized* by the momentum p . That is, the momentum eigenstates form a large family of orthogonal states distinguished by a particular value of p . This is the interpretation we will adopt for the role of the momentum in the generalized Kodama state. To see this explicitly, we rewrite the state (21) in a more suggestive form by absorbing irrelevant factors which depend only on the momentum through $\Gamma[E]$ into the normalization constant. The state becomes:

$$\Psi_R[A] = \mathcal{P} \exp \left[i\kappa \int_{\Sigma} A \wedge R - \frac{1}{2(1+\beta^2)} Y_{CS}[A] \right]. \quad (24)$$

Here we see explicitly, A plays the role of the position variable x , the Levi-Civita curvature $R = d\Gamma + \Gamma \wedge \Gamma$ plays the role of the momentum, $\kappa = \frac{3(1+\beta^2)}{2k\lambda\beta^3}$ is simply a scaling factor, we have a dimensionless energy $\frac{1}{2(1+\beta^2)}$, and the Chern-Simons term $\int Y_{CS}[A]$ plays the role of the time variable. We note that it has been independently suggested that the Chern-Simons invariant is a natural time variable on the canonical phase space[9]. With this interpretation, the generalized state is not a single state at all, but a large class of states parameterized by a specific configuration of the three-dimensional Levi-Civita curvature, R .

3.2 The naive inner product

We can push the analogy further by considering the inner product between two states with different curvature configurations $\langle \Psi_{R'} | \Psi_R \rangle$. The analogue of this is the inner product of two momentum states:

$$\begin{aligned} \langle p' | p \rangle &= \int d^n x \, \Psi_{p'}^*[x, t] \Psi_p[x, t] \\ &= \mathcal{P}[p', p] \int d^n x \, \exp[-i(p' - p) \cdot x] \\ &\sim \delta^n(p' - p). \end{aligned} \tag{25}$$

Following along these lines, we define a naive inner product:

$$\begin{aligned} \langle \Psi_{R'} | \Psi_R \rangle_{naive} &= \mathcal{P}[\Gamma', \Gamma] \int_{\Sigma} \mathcal{D}A \, \Psi_{R'}^*[A] \Psi_R[A] \\ &= \int \mathcal{D}A \, \exp \left[-i\kappa \int_{\Sigma} A \wedge (R' - R) \right] \end{aligned} \tag{26}$$

Formally integrating over the space of connections we have

$$\langle \Psi_{R'} | \Psi_R \rangle_{naive} \sim \delta(R' - R). \tag{27}$$

Thus, under this naive inner product, two states are orthogonal unless they are parameterized by the same configuration of the Levi-Civita curvature. The deficiency of this inner product is that it is not gauge invariant. If the two fields R' and R represent the same curvature written in a different gauge, either $SU(2)$ or diffeomorphism, they will be orthogonal. Thus, we need to modify the inner product to make it gauge invariant.

3.3 Gauge covariance and the kinematical inner product

In order to define a gauge invariant inner product we first need to discuss the gauge properties of the generalized states. The set of states Ψ_R are not strictly speaking $SU(2)$ gauge or diffeomorphism invariant. The reason is because of the presence of the parameter R in the argument which acts like an effective “background” against which one can measure the effect of a gauge transformation or diffeomorphism. This is not unfamiliar. We encounter

the same difficulty with the spin-network states where the graph serves as a “background” against which one can measure the effect of a diffeomorphism shifting the connection, A (see e.g. [15]). In the spin network states, the action of a one-parameter diffeomorphism $\phi_{-\bar{N}}$ on the connection configuration, A , is equivalent to shifting the graph in the opposite direction by $\phi_{\bar{N}}$. Similarly, one can show that the combined effect of an $SU(2)$ gauge transformation and diffeomorphism on the field configuration which we will denote by $\phi_{\{g^{-1}, -\bar{N}\}}A$ is equivalent to the inverse transformation on the curvature denoted by $\phi_{\{g, \bar{N}\}}R$. Thus, under the action of the Gauss and diffeomorphism constraint, the state transforms as follows:

$$\Psi_R \rightarrow \hat{U}_\phi(g^{-1}, -\bar{N})\Psi_R = \Psi_{\phi_{\{g, \bar{N}\}}R}. \quad (28)$$

The strategy with the spin network states is to implement the diffeomorphism symmetry via the inner product where the diffeomorphism symmetry is manageable, and this is the strategy we will also adopt. To make the inner product gauge invariant, we introduce the measure $\mathcal{D}\phi_{\{g, \bar{N}\}}$ over the set of all $SU(2)$ gauge transformations (which may be accomplished by the Haar measure) and the set of all diffeomorphisms. Although a measure over the set of all diffeomorphisms is undefined, the end result may still be manageable due to the specific form of the integrand. This is true in the inner product on spin network states, where the problem of defining a measure over the group of diffeomorphisms is relegated to the problem of determining when two graphs are in the same equivalence class of knots. A similar result applies here. To see this, we define the kinematical inner product as follows:

$$\langle \Psi_{R'} | \Psi_R \rangle_{kin} = \int \mathcal{D}\phi_{\{g, \bar{N}\}} \langle \Psi_{R'} | U_\phi(g, \bar{N}) \Psi_R \rangle_{naive}. \quad (29)$$

From the gauge covariance of the states Ψ_R we have:

$$\begin{aligned} \langle \Psi_{R'} | \Psi_R \rangle_{kin} &= \int \mathcal{D}\phi_{\{g, \bar{N}\}} \langle \hat{\phi}_{\{g^{-1}, -\bar{N}\}} \Psi_{R'} | \Psi_R \rangle \\ &= \int \mathcal{D}\phi_{\{g, \bar{N}\}} \langle \Psi_{\phi_{\{g, \bar{N}\}}R'} | \Psi_R \rangle \\ &\sim \int \mathcal{D}\phi_{\{g, \bar{N}\}} \delta(\phi_{\{g, \bar{N}\}}R' - R) \\ &= \delta(\mathcal{R}' - \mathcal{R}) \end{aligned} \quad (30)$$

where in the last line \mathcal{R}' and \mathcal{R} are elements of the equivalence class of curvatures modulo $SU(2)$ -gauge and diffeomorphism transformations. Thus,

the problem of defining a measure over the set of diffeomorphisms is reduced to the problem of determining when two curvatures are gauge related—a problem that is all too familiar from classical General Relativity. The states Ψ_R and $\Psi_{R'}$ are orthogonal unless there is a diffeomorphism and/or $SU(2)$ gauge transformation relating R and R' .

3.4 Levi-Civita curvature operator

Continuing the analogy with the momentum eigenstates we proceed to define a Levi-Civita curvature operator. We recall the momentum operator can be defined in terms of the momentum eigenstates:

$$\hat{p} = \int d^n p' p' |p'\rangle \langle p'|. \quad (31)$$

By construction, the states $|p\rangle$ are then eigenstates of \hat{p} .

Since the generalized states $|\Psi_R\rangle$ represent a family of orthogonal states parameterized by the curvature configuration R , it is natural to define a curvature operator such that the states are curvature eigenstates. Analogous to the momentum operator, we define the operator in its diagonal form as follows (writing $\phi = \phi_{\{g, \bar{N}\}}$):

$$\int_{\Sigma} \alpha \wedge \hat{R}_{\Gamma} = \int \mathcal{D}\phi \mathcal{D}\Gamma' \left[\left(\int_{\Sigma} \lambda \wedge \phi R'_{\Gamma'} \right) |\Psi_{\phi R'}\rangle \langle \Psi_{\phi R'}| \right] \quad (32)$$

where α is an arbitrary $su(2)$ valued one-form serving as a test function, and $\mathcal{D}\Gamma'$ is an appropriate measure to integrate over all values of the Levi-Civita 3-curvature $R'_{\Gamma'}$. When operating on a state $|\Psi_R\rangle$ it is understood that intermediate inner product is the naive inner product. That is,

$$\begin{aligned} & \int_{\Sigma} \alpha \wedge \hat{R} |\Psi_R\rangle \\ &= \int \mathcal{D}\phi \mathcal{D}\Gamma' \left[\left(\int_{\Sigma} \alpha \wedge \phi R'_{\Gamma'} \right) |\Psi_{\phi R'}\rangle \langle \Psi_{\phi R'} | \Psi_R \rangle_{naive} \right] \\ &= \int \mathcal{D}\phi \mathcal{D}\Gamma' \left[\delta(\phi R' - R) \left(\int_{\Sigma} \alpha \wedge \phi R'_{\Gamma'} \right) |\Psi_{\phi R'}\rangle \right] \\ &= \int_{\Sigma} \alpha \wedge R_{\Gamma} |\Psi_R\rangle \end{aligned} \quad (33)$$

Thus, with this definition, the states $|\Psi_R\rangle$ are eigenstates of the curvature operator \hat{R}_Γ :

$$\int_\Sigma \alpha \wedge \hat{R} |\Psi_R\rangle = \int_\Sigma \alpha \wedge R |\Psi_R\rangle. \quad (34)$$

3.5 The Hamiltonian constraint

We now address the issue of the Hamiltonian constraint. The beauty of the complex Ashtekar formalism is that the Hamiltonian constraint simplifies to the point where it is solvable, admitting the Kodama state as a quantum solution to the Hamiltonian constraint. Our partial parity violating version of the Ashtekar action held the promise of a simplified Hamiltonian until the reality constraints were imposed, which introduced second class constraints on the torsion. When solved, the constraint implies that the left and right handed connections no longer commute because they both contain a term $\Gamma[E]$. The real formulation of the Holst action, is plagued with the same problem. Although the phase space consists of just the connection A and its conjugate momentum, the Hamiltonian constraint explicitly contains terms involving $\Gamma[E]$. Depending on how one writes the constraint, they enter via extrinsic curvature terms, $K = \frac{1}{\beta}(\Gamma - A)$, or through the Levi-Civita curvature, $R = d\Gamma + \Gamma \wedge \Gamma$. The standard representation of the Hamiltonian constraint is⁶

$$C_H = \int_\Sigma * \Sigma \wedge \left(F + (1 + \beta^2) \left(\frac{1}{\beta^2} D_\Gamma K - K \wedge K \right) - \frac{\lambda}{3} \Sigma \right), \quad (35)$$

where $F = F[A]$ is the curvature of A . Because of the complexity of this constraint, it appears to be very difficult to determine if our generalized Kodama states are in the kernel of the corresponding quantum operator. However, the constraint can be rewritten by substituting the extrinsic curvature terms in favor of the Levi-Civita curvature. The constraint then takes the form

$$C_H = \int_\Sigma * \Sigma \wedge \left((1 + \frac{1}{\beta^2}) R - \frac{1}{\beta^2} F - \frac{\lambda}{3} \Sigma \right). \quad (36)$$

This form is particularly convenient for our purposes because we have already suggested a form for the Levi-Civita curvature operator. In the standard

⁶The term involving $*\Sigma \wedge D_\Gamma K$ may be unfamiliar since it is usually not included in the constraint. The term does explicitly occur in the Hamiltonian decomposition, but it can be integrated away using the fact that Γ is torsion free so $D_\Gamma * \Sigma = 0$. We will keep the term explicitly because it simplifies the algebra in the next step.

Kodama operator ordering where $*\Sigma$ is placed on the far left, the full set of generalized Kodama states are in the kernel of the Hamiltonian by virtue of being in the kernel of the quantum operator

$$\int_{\Sigma} \alpha \wedge \left(\left(1 + \frac{1}{\beta^2}\right) \hat{R} - \frac{1}{\beta^2} \hat{F} - \frac{\lambda}{3} \hat{\Sigma} \right) \quad (37)$$

where α is a test function. To see this, in the connection representation Σ is a differential operator which acts on $\Psi_R[A]$ by:

$$-\frac{\lambda}{3} \Sigma \Psi_R[A] = i2k\beta \frac{\lambda}{3} \frac{\delta \Psi_R[A]}{\delta A} = \left(\frac{1}{\beta^2} F - \left(1 + \frac{1}{\beta^2}\right) R \right) \Psi_R[A] . \quad (38)$$

The curvature F cancels since \hat{F} is multiplicative in the connection representation. We are left with

$$\left(1 + \frac{1}{\beta^2}\right) \int_{\Sigma} \alpha \wedge (\hat{R} - R) \Psi_R[A] , \quad (39)$$

which vanishes by (34). Thus, for any curvature configuration, R , with the standard Kodama operator ordering we have

$$\hat{C}_H |\Psi_R\rangle = 0. \quad (40)$$

4 Concluding Remarks

We have shown that the Kodama state can be generalized to real values of the Immirzi parameter, and the generalization appears to open up a large class of physical states. In the connection representation they are the exponent of the Chern-Simons invariant together with an extra term, so we might expect that in the spin network basis they may be expressed as a generalization of the Kauffman bracket. The states share many properties in common with the momentum eigenstates when the Levi-Civita is identified with the “momentum” parameterizing the family of states. Following this analogy, we have shown that the generalized Kodama states are eigenstates of a naturally defined Levi-Civita curvature operator with eigenvalues given by the curvature configuration parameterizing the state. This definition of the curvature operator places the full sector in the physical Hilbert space. Naturally this operator must withstand a program of consistency checks to verify its viability as the Levi-Civita curvature operator, but we hope that we have

laid the groundwork to begin such a program. We set out to generalize the Kodama state in an attempt to resolve some of the known issues associated with the original version. We have shown that our generalization solves two of the known problems: reality conditions, and normalizability. The problem of defining the reality conditions does not exist in the real theory, and we have shown that the states are delta-function normalizable under a natural inner-product. In the second paper of this two paper series we will show that the states are CPT invariant, and, by fine tuning of the coupling constants, they can be made to be invariant under large transformations. In addition, we will discuss the physical interpretation of the states.

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